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Preserving chaos: Control strategies to preserve complex dynamics with potential relevance to biological disorders

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This paper considers the situation in which an originally chaotic orbit would, in the absence of intervention, become periodic as a result of slow system drift through a bifurcation. In the biological context, such a bifurcation is often undesirable: there are many cases, occurring in a wide variety of different situations, where loss of complexity and the emergence of periodicity are associated with pathology (such situations have been called "dynamical disease"). Motivated by this, we investigate the possibility of using small control perturbations to preserve chaotic motion past the point where it would otherwise bifurcate to periodicity.

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I. INTRODUCTION

A. Preserving chaos

In this paper we study the following situation. There is a nonlinear system that behaves chaotically when a system parameter p is below some critical value, $p<p_c$. As p increases through p_c, there is a bifurcation in which the chaotic motion is replaced by periodic motion. The problem we address is whether, by using small controls, it is possible to keep the motion chaotic when $p>p_c$. As an example, consider the case of a d-dimensional map,

$$x_{n+1}=\tilde{F}(x_n,p,c_n) , \quad (1)$$

where x_n is the d-dimensional state of the system at time n, and c_n denotes the control variable. Roughly, we can think of the control as tuning some "knob" whose setting c_n is at our disposal.

For $c_n\equiv 0$, the map (1) bifurcates from chaos to periodicity as p increases through p_c. The question is how do we program the time dependence of c_n to ensure chaotic dynamics when $p>p_c$. We desire, in addition, that $|c_n|$ be small, and that the control only need to be applied infrequently (i.e., $c_n=0$ most of time).

Past work on controlling chaos has considered the situation where the motion is chaotic and one wishes to modify it to obtain improved system performance by stabilizing a chosen unstable periodic orbit embedded in the chaotic attractor [1,2]. In the present work, however, we regard the complex temporal behavior of the chaotic orbit as "good." Thus we wish to intervene in such a way as to keep it alive in situations where it would naturally be absent. The motivation for this is biological, and is discussed in the next subsection.

B. Biological motivation

It has been known since the 1940's that systems of nonlinear differential equations representative of self-excitatory biological processes, such as the heart beat and neuronal discharge, manifest parameter regions that generate chaotic orbits [3]. In the 1970s, the chaotic behavior of discrete maps also became relevant to biological systems such as population dynamics within ecological context in regimes of high fecundity and nutritional support [4]. Other work suggested that patterns of behavior in biological observables over time better represented some medical diseases than did characteristic structural and/or chemical abnormalities; that bifurcations between dynamical states could be associated with the onset and pathophysiology of a variety of disease states [5]. Since that time, one particularly counterintuitive (and somewhat controversial) theme of a number of studies carried out within the context of "dynamical diseases" is that in some biomedical systems, chaotic dynamical behavior is "normal." Bifurcations to periodic behavior (observed directly or reflected in decreased statistical measures of complexity) are viewed as a pathophysiological loss of the range of adaptive possibilities. The idea was that "getting stuck" can lead to disease states.

A diverse list of examples of emergent pathophysiolog-

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ical periodicity (and/or decreasing measure indicative of dynamical complexity) in otherwise more irregular, normal time dynamics includes cell counts in hematological disorders [6], stimulant drug induced abnormalities in patterns in time of the behavior of brain enzymes, receptors, and animal explorations of space [7], cardiac interbeat interval patterns in a variety of cardiac disorders [8], the resting record in a variety of signal sensitive biological systems following desensitization [9], experimental epilepsy [10], hormone release patterns correlated with the spontaneous mutation of a neuroendocrine cell to a neoplastic tumor [11], the prediction of immunological rejection of heart transplants [12], the electroencephalographic behavior of the human brain in the presence of neurodegenerative disorders [13], neuroendocrine, cardiac, and electroencephalographic changes with aging [14], and imminent ventricular fibrillation in human subjects [15].

The above references are only a representative list, and it is possible to cite many more illustrating the general thesis that loss of chaos is often associated with disease [16]. Thus we can view the onset of a diseased state as being caused by a slow drift of the parameter p in Eq. (1) through the critical value p_c. As an example suggesting the utility of control in some of these situations we note the recent work [17] that has the goal of preventing epileptic seizures by time dependent stimuli applied to an appropriate region of the brain.

C. Loss regions and control strategy

The three most common bifurcations that can lead from chaotic motion directly to a low period attracting periodic orbit are [18] (1) crises [19], (2) type I intermittency [20], and (3) type III intermittency [20]. In all of the above three cases, for $p>p_c$, one can identify a loss region L, such that, after the orbit falls in L, it is rapidly drawn to the periodic orbit. Before falling in L the orbit motion can exhibit characteristic chaoticlike behavior; that is, it is a chaotic transient.

One strategy to ensure chaos for $p>p_c$ is to consider successive preiterates of L,

$$L_1=\tilde{F}^{-1}(L,p,0)\ ,$$
$$L_2=\tilde{F}^{-1}(L_1,p,0)=\tilde{F}^{-2}(L,p,0)\ ,$$
$$L_3=\tilde{F}^{-1}(L_2,p,0)=\tilde{F}^{-3}(L,p,0)\ ,$$
$$\vdots$$
$$L_m=\tilde{F}^{-1}(L_{m-1},p,0)=\tilde{F}^{-m}(L,p,0)\ .$$

Thus L_m is the set of points that map to the loss region L in m iterates. Note that as m increases, the width of L_m in the unstable direction (or directions) has a general tendency to shrink exponentially. (There is always at least one unstable direction due to the existence of chaos.) This suggests the following approach. Pick some suitable value of m, which we denote M. Assume that the orbit initially starts outside the regions $L_{M+1}\cup L_M\cup\cdots\cup L_1\cup L$. If the orbit lands in L_{M+1} on iterate n we apply our control c_n so as to kick the orbit out of L_M on the next iterate. Because L_M is thin, the required c_n can be small. By making M larger we typically make the required size of the control smaller. Thus there is a trade off involving the control size and the number of preiterates M. Consideration of this trade off becomes particularly important if noise is present or if the dynamical system F is not known with absolute precision. After the orbit is kicked out of L_M, it is expected that it will execute a chaotic orbit, until it again falls in L_{M+1}, at which time the small control is again activated, and so on. Thus we achieve the desired result that the control can be set to zero much of the time.

We call the above strategy A. There are also other strategies that can be usefully formulated. These, however, are less general and depend on the type of bifurcation involved (i.e., crisis, type I intermittency, or type III intermittency). We consider some of these other strategies and compare them to strategy A in the subsequent discussions dealing with the relevant bifurcations [21].

In Sec. II, we illustrate our control strategies using one-dimensional map examples exhibiting crisis, type I intermittency, and type III intermittency transitions from chaos to periodicity. Sec. III considers a two-dimensional map example, and Sec. IV focuses on the effect of noise.

We emphasize that, although our discussion is within the context of maps [Eq. (1)], the ideas are also relevant to continuous time dynamical systems (e.g., ordinary differential equations).

II. ONE-DIMENSIONAL MAP EXAMPLES

A. Crisis

We considered the situation shown in Fig. 1 for $c_n\equiv0$. When $p<p_c$, there is a chaotic attractor in the region $0\le x\le 1$, and a coexisting attracting periodic orbit of period one located at $x=x_p<0$; see Fig. 1(a). As p increases through p_c the chaotic attractor is destroyed by a crisis [Fig. 1(b)], and almost all points with respect to the Lebesgue measure that are initialized in $0\le x\le 1$ eventually go to the periodic attractor. We identify the loss region L as shown in Fig. 1(c). An orbit point in L maps one iterate later to the interval L_+, and one iterate after that to the region $x<0$, after which it approaches the periodic orbit $x=x_p$ monotonically. Also shown in Fig. 1(c) is the set L_1 of points that map to L on one iterate. Each set L_m consists of 2^m intervals; thus, as shown in Fig. 1(c), the set L_1 consists of two intervals. As a numerical example, assume that the map applying for $x>0$ is of the logistic form, and that the dependence of $\tilde{F}(x_n,p,c_n)$ on c_n is simply additive,

$$\tilde{F}(x_n,p,c_n)=F(x_n,p)+c_n=px_n(1-x_n)+c_n\ . \tag{2}$$

The critical value of p for Eq. (2) (with $c_n\equiv0$) is $p_c=4$. For $p>p_c$ the width of the loss region L is

$$w(L)=[(p-p_c)/p_c]^{1/2}(p_c/p)^{1/2}\ , \tag{3a}$$

and the width of L_+ is

$$w(L_+)=(p-p_c)/p_c\ . \tag{3b}$$

To apply strategy A the control c_n is programmed as follows. Suppose that at time n the orbital point x_n falls

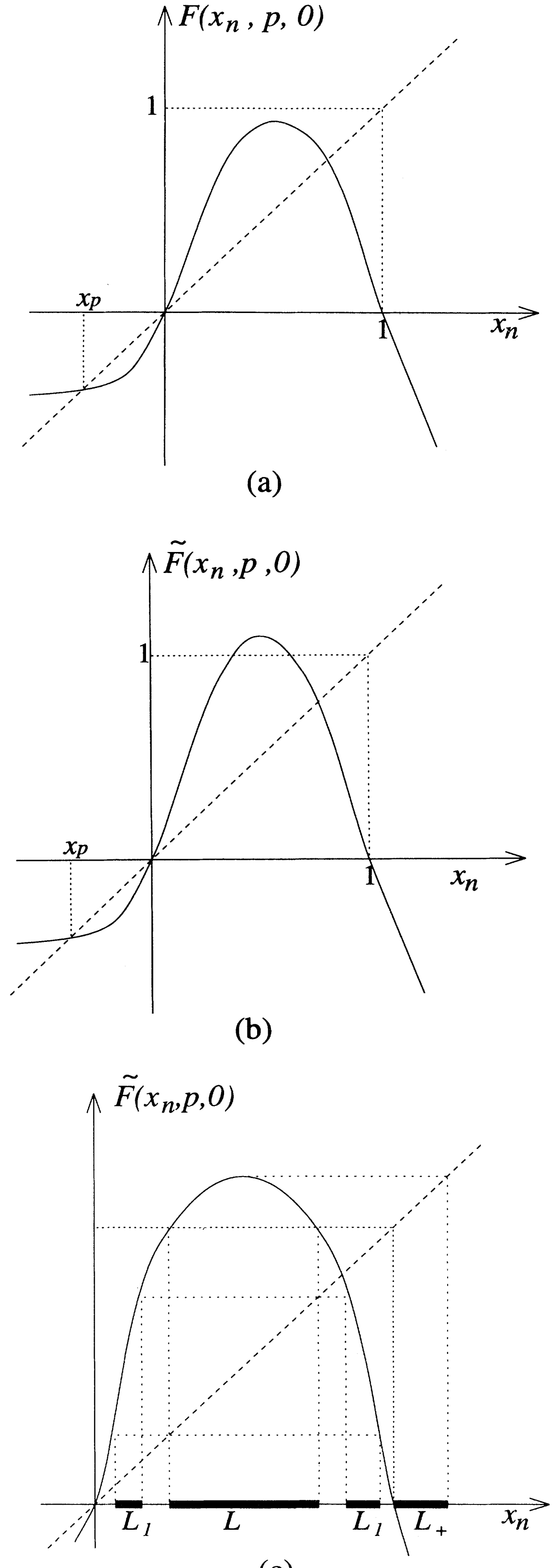

FIG. 1. Illustration of a crisis transition from a chaotic attractor to periodic motion ($x = x_p$).

in one of the 2^{M+1} intervals of L_{M+1} and denote this interval $[x_l^{M+1}, x_r^{M+1}]$. Assume without loss of generality that $F(x_l^{M+1}, p, 0) < F(x_r^{M+1}, p, 0)$. We desire $x_{n+1} < F(x_l^{M+1}, p, 0)$ or $x_{n+1} > F(x_r^{M+1}, p, 0)$. To do so we choose c_n to be whichever of the following two values has the smaller magnitude,

$$c_n = \begin{cases} 1.1[px_l^{M+1}(1-x_l^{M+1}) - px_n(1-x_n)] , \\ 1.1[px_r^{M+1}(1-x_r^{M+1}) - px_n(1-x_n)] . \end{cases} \tag{4}$$

Figure 2 shows a plot of x_n versus n with strategy A control for the case of $p = 4.1$ and $M = 1$. Figure 3(a) shows c_n versus n for the same case. Over the time period shown in Fig. 3(a), the largest value of $|c_n|$ is $c_{\max} = 0.029$, and the fraction of the time with $c_n \neq 0$ is 0.085. Increasing M to $M = 5$, we obtain the plot c_n versus n in Fig. 3(b). By increasing M the largest value of $|c_n|$ is reduced to $c_{\max} = 0.0017$ in Fig. 3(b).

We now consider a very simple alternate strategy that works particularly well when p is only slightly bigger than p_c. We note from Eqs. (3) that when p is close to p_c, $w(L_+)$ is smaller than $w(L)$ by the factor $[(p-p_c)/p_c]^{1/2}$. Thus it is easier to kick the orbit out of L_+ than L. Strategy A reduces the kick by making M large enough. Instead, in our alternative strategy we wait until the orbit falls in L, and then apply a c_n so that we kick the orbit from L_+ into the region $x < 1$. This can be accomplished, for example, by setting

$$c_n = -(p - p_c)/p_c \tag{5}$$

whenever the orbit falls in L. This has the effect of changing the map from the form shown in Fig. 1(c) to that shown in Fig. 4. Using the same value of p as in Fig. 3(a) and 3(b), the alternate strategy yields the plot of c_n versus n in Fig. 3(c). Note that $c_{\max}$ for strategy A with $M = 1$ [Fig. 3(a)] is larger than the value given by Eq. (5) [$c_{\max} = 0.029$, as compared to $(p - p_c)/p_c = 0.025$], while, when M is increased to $M = 5$ [Fig. 3(b)], strategy A yields a much lower value.

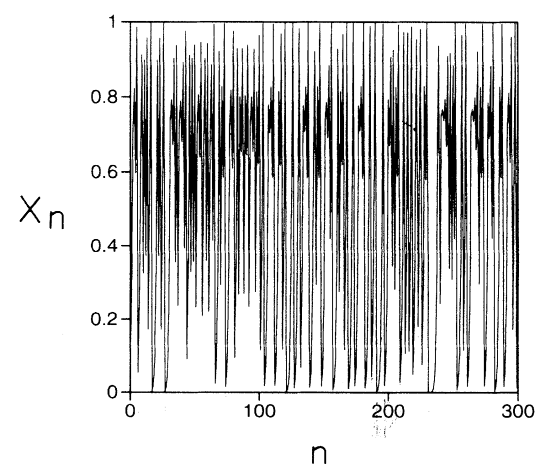

FIG. 2. x_n versus n with control for $M = 1$ and $p = 4.1 > p_c = 4$.

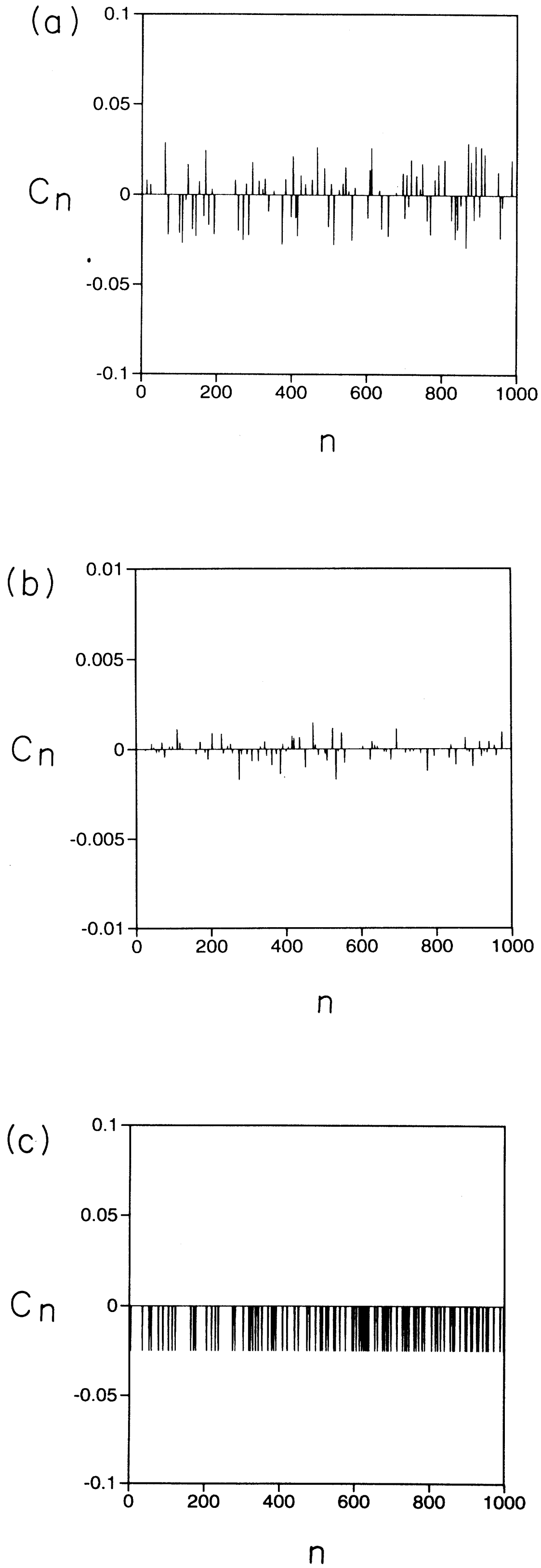

FIG. 3. c_n versus n for $p=4.1$. (a) Strategy A with $M=1$, (b) strategy A with $M=5$, and (c) the alternate strategy.

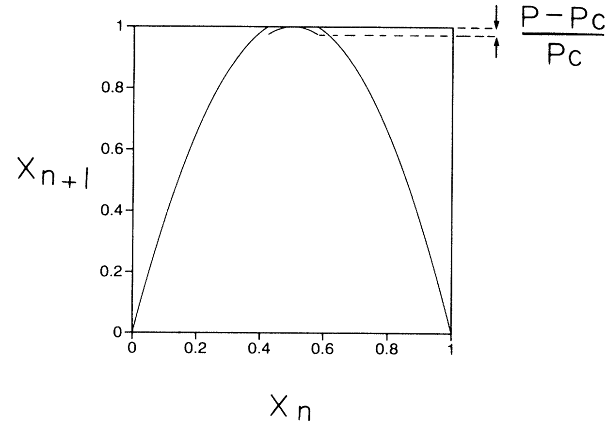

FIG. 4. Effective one-dimensional map associated with the alternate control strategy given by (4) whenever x_n is in L.

B. Type I intermittency

The example we consider is

$$\tilde{F}(x_n,p,c_n)=F(x_n,p)+c_n$$
$$=4px_n(1-x_n)[1-px_n(1-x_n)]+c_n\ . \tag{6}$$

Figure 5(a) shows a plot of $F(x_n,p)$ versus x_n for $p=3.350<p_c=3.375$. In this case there is a chaotic attractor located in $0\leq x\leq 1$. As p increases through p_c, a tangent bifurcation occurs creating a stable attracting period one orbit $x=x_A$ and a repelling period one orbit $x=x_R$ (see Fig. 6). Figure 5(b) shows a plot of $F(x_n,p)$ versus x_n for $p=3.4>p_c$. (The points x_A and x_R are the new intersections of the diagonal dashed line with F.) A bifurcation diagram for this map is shown in Fig. 7.

Figure 6 shows a schematic identifying the loss region L for this case. Here L is the interval $(1-x_R)<x<x_R$ [$(1-x_R)$ is a preiterate of x_R]. Note that, near the bifurcation (i.e., when $p-p_c>0$ is small), x_A is close to x_R; $x_A=x_R$ at $p=p_c$.

Figure 8(a) shows the orbit x_n versus n for a case without control for $p=3.5>p_c$. Figure 8(b) shows the orbit from the same initial condition as Fig. 8(a) using strategy A control with $M=2$, and Fig. 8(c) shows the resulting c_n versus n. The fraction of the time that the control is activated is 0.08, and the largest $|c_n|$ over the time interval plotted is $c_{\max}=0.01$.

Unlike the crisis case, for type I intermittency the width of L does not approach zero as $p-p_c\rightarrow 0^+$. Rather, L appears immediately at $p=p_c$ with nonzero width. On the other hand, for small $p-p_c>0$, the distance (x_R-x_A) is small, namely, $(x_R-x_A)\sim O(\sqrt{p-p_c})$. This motivates the following alternate strategy applicable for p close to p_c. This alternate strategy utilizes a control of order $\sqrt{p-p_c}$ and hence is small if p is close to p_c. If x_n first enters L in the region $x_A<x<x_R$, we apply a positive kick c_n to place x_{n+1} in the region $x>x_R$. Once in this region the orbit is repelled from x_R, moves to the right, and executes a chaotic transient orbit, until it again

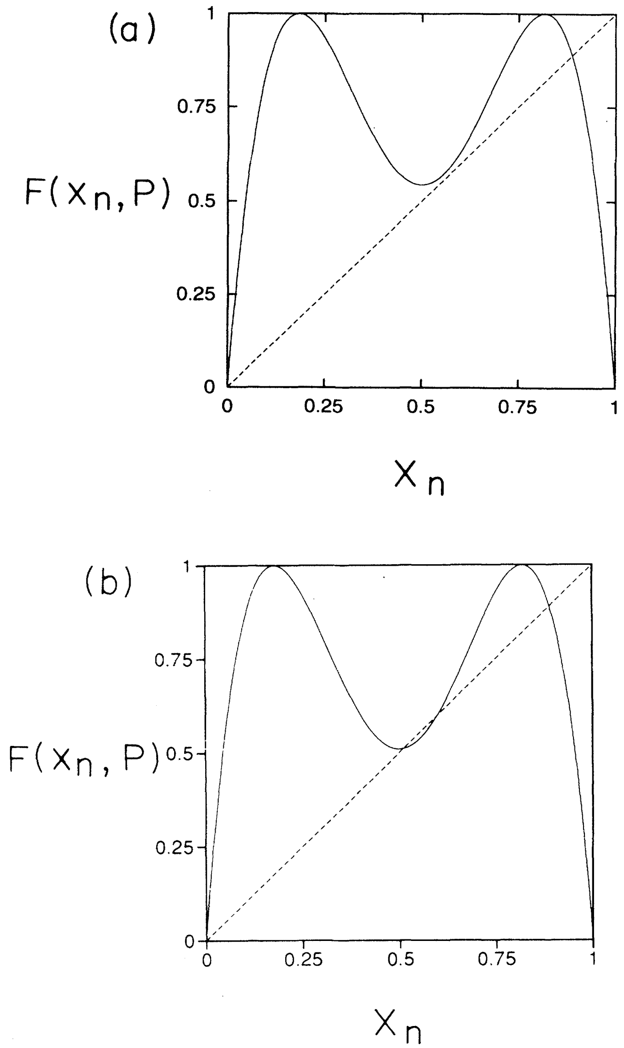

FIG. 5. $F(x_n,p)$ versus x_n for (a) $p=3.35<p_c=3.375$ and (b) $p=3.40>p_c$.

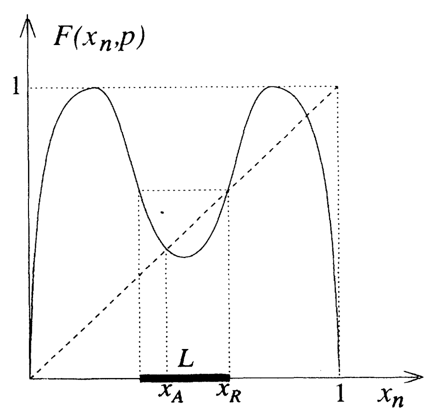

FIG. 6. Schematic identifying the loss region L.

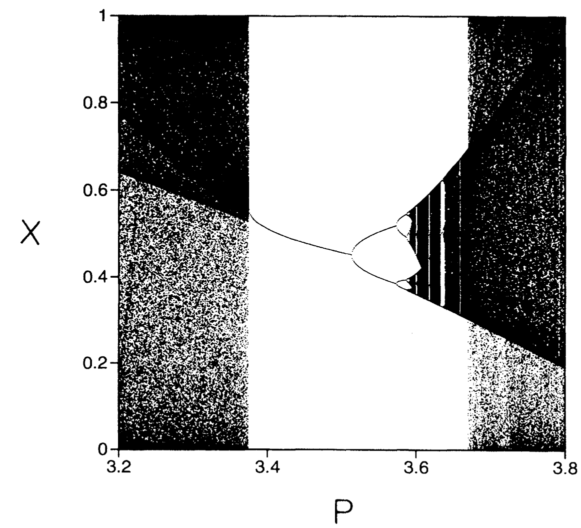

FIG. 7. Bifurcation diagram for $F(x_n,p)$. Note the intermittent transition at $p=p_c=3.375$.

enters L. If x_n first enters L in the region $x<x_A$ with x relatively far from x_A [i.e., $(x_A-x)>O(\sqrt{p-p_c})$], we can wait until it either lands in $x_A<x<x_R$ or else approaches sufficiently close to x_A with $x\leq x_A$. In either case, a kick of order $\sqrt{p-p_c}$ will be capable of placing a next iterate in $x>x_R$.

C. Type III intermittency

We consider the map

$$\tilde{F}(x_n,p,c_n)=F(x_n,p)+c_n$$
$$=x_n[p(12.5-7p)x_n^4-p(11.5-7p)x_n^2+x_n^2-1]+c_n\ . \qquad (7)$$

The map $F(x_n,p)$ has a period one orbit at $x=0$. As p increases through $p_c=1$, this period one orbit becomes unstable, experiencing an inverse period doubling bifurcation. The map yields a chaotic attractor for $p>p_c$. (Note that the role of parameter increase here is opposite to the convention used in the previous discussion.) Figures 9(a) and 9(b) show plots of the map $F(x_n,p)$ for $p=1.2>p_c$ (chaotic attractor) and for $p=0.8<p_c$ ($x=0$ being the periodic attractor). We indicate that loss region in Fig. 9(b) as the interval bounded by the two components of the unstable period two orbit created at the bifurcation. For small $(p_c-p)>0$ the width of L scales as $w(L)\sim O(\sqrt{p_c-p}\ )$. (This is similar to the crisis case.) Once an orbit enters L its distance to the period one attractor at $x=0$ decreases monotonically as the orbit flips alternatively from $x>0$ to $x<0$. Also indicated in Fig. 9(b) are the intervals comprising L_1.

Figure 10(a) shows the orbit x_n versus n for a case without control for $p=0.9<p_c$. Figure 10(b) shows the

orbit from the same initial condition as Fig. 10(a) using strategy A control with $M=2$. Figure 10(c) shows the resulting c_n versus n. The fraction of time the control is activated is 0.03, and the largest $|c_n|$ over the time interval of the plot is $c_{\max}=0.016$.

III. TWO-DIMENSIONAL MAP EXAMPLE

Strategy A can also be applied without the explicit construction of all the L_m. This is particularly useful in the case of two-dimensional maps. Consider the crisis situation schematically illustrated in Figs. 11. For $p<p_c$ there is an chaotic attractor [Fig. 11(a)]. The point A is an unstable saddle on the basin boundary which is the closure of the stable manifold of A. As p increases

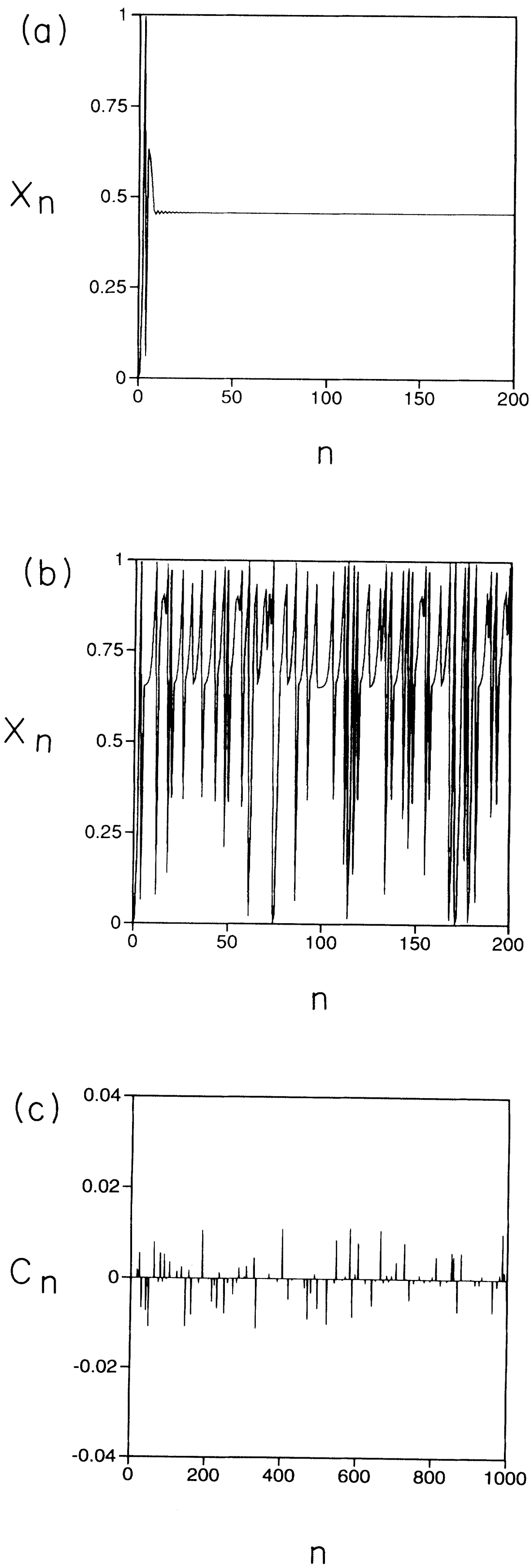

FIG 8. (a) x_n versus n without control for $p=3.5>p_c$. (b) x_n versus n with control and $M=2$ for $p=3.5>p_c$. (c) c_n versus n for the conditions of (b).

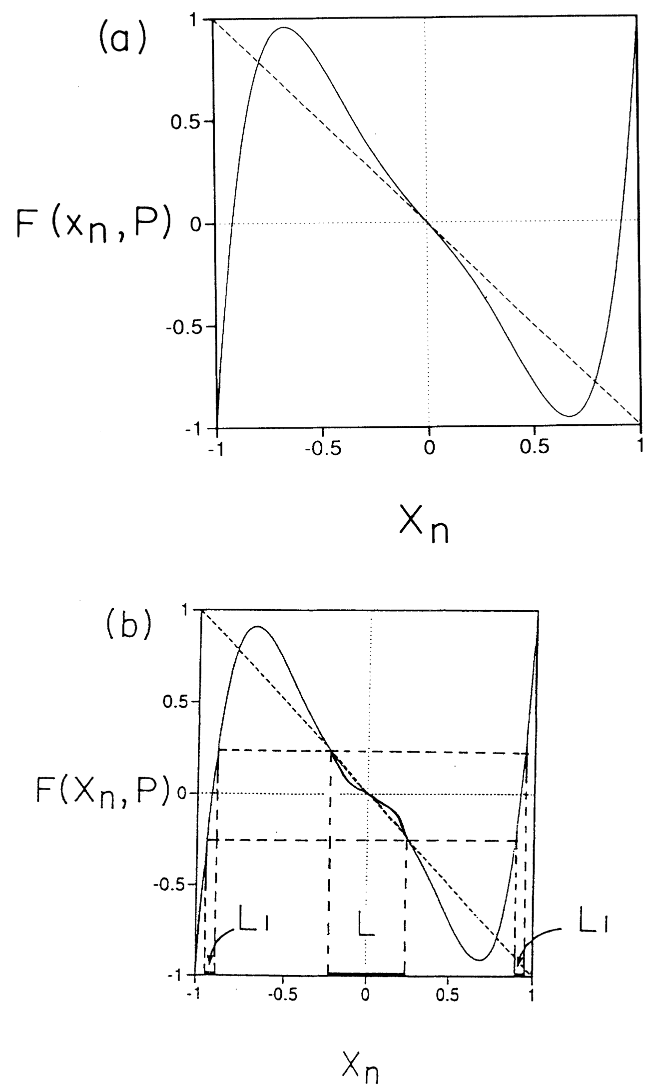

FIG. 9. $F(x_n,p)$ versus x_n for (a) $p=1.2>p_c=1.0$ and (b) $p=0.8<p_c$ (the curvature near the origin is exaggerated for clarity).

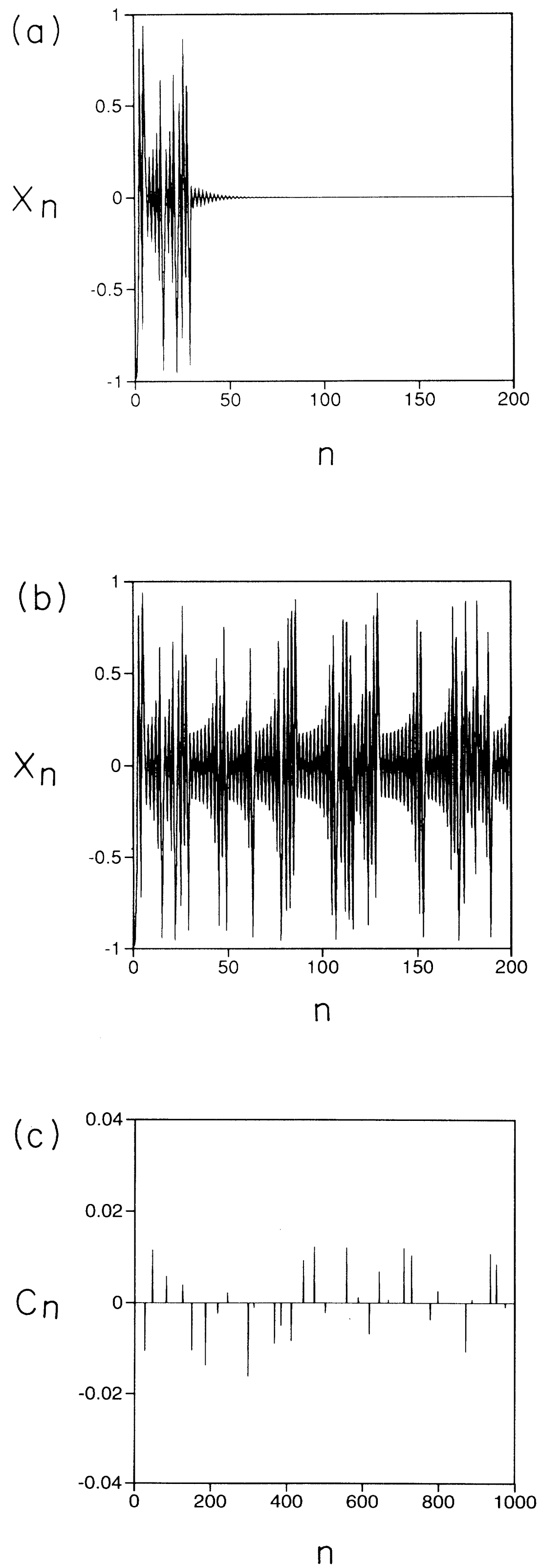

FIG. 10. Results of $p=0.9<p_c$. (a) x_n versus n without control. (b) x_n versus n with strategy A control and $M=2$. (c) c_n versus n.

through p_c, the attractor is destroyed by colliding with the basin boundary, resulting in a situation shown in Fig. 11(b) [22]. We assume that after a chaotic transient almost all points go to a periodic attractor located elsewhere in the phase space. We call the shaded region in Fig. 11(b) L_+ according to the terminology in Sec. II A. Imagine a point x in the former basin of attraction, and examine the $(M+2)$th iterate of x. If this $(M+2)$th iterate falls in L_+, then we say x is in L_{M+1} and apply a kick c_n to the system so that the next iterate of x falls outside of L_M. [By this we mean that the $(M+1)$th iterate of the kicked point without further perturbation falls outside of L_+.] After this we set $c_n=0$ and wait until the procedure needs to be repeated again.

In practice, the above steps can be further simplified. Specifically, we can replace the segment of the stable manifold near the shaded area in Fig. 11(b) by a straight line such that L_+ is to its right. Instead of considering the point falling into the shaded region we can just test whether it is on the right side of this straight line.

Now we use the Hénon map

$$x_{n+1}=p-x_n^2+0.3y_n+c_n ,$$

$$y_{n+1}=x_n ,$$

as our numerical example to illustrate these ideas. Here $p_c \approx 1.42$ and the straight line mentioned above is chosen to be $10x-y=20$ for $p=2.0$. The value of c_n picked has

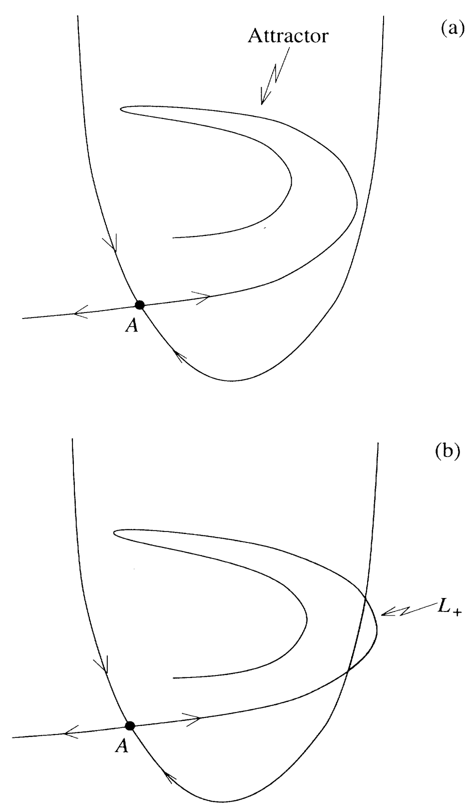

FIG. 11. Schematic of a two-dimensional crisis transition. (a) $p<p_c$ and (b) $p>p_c$.

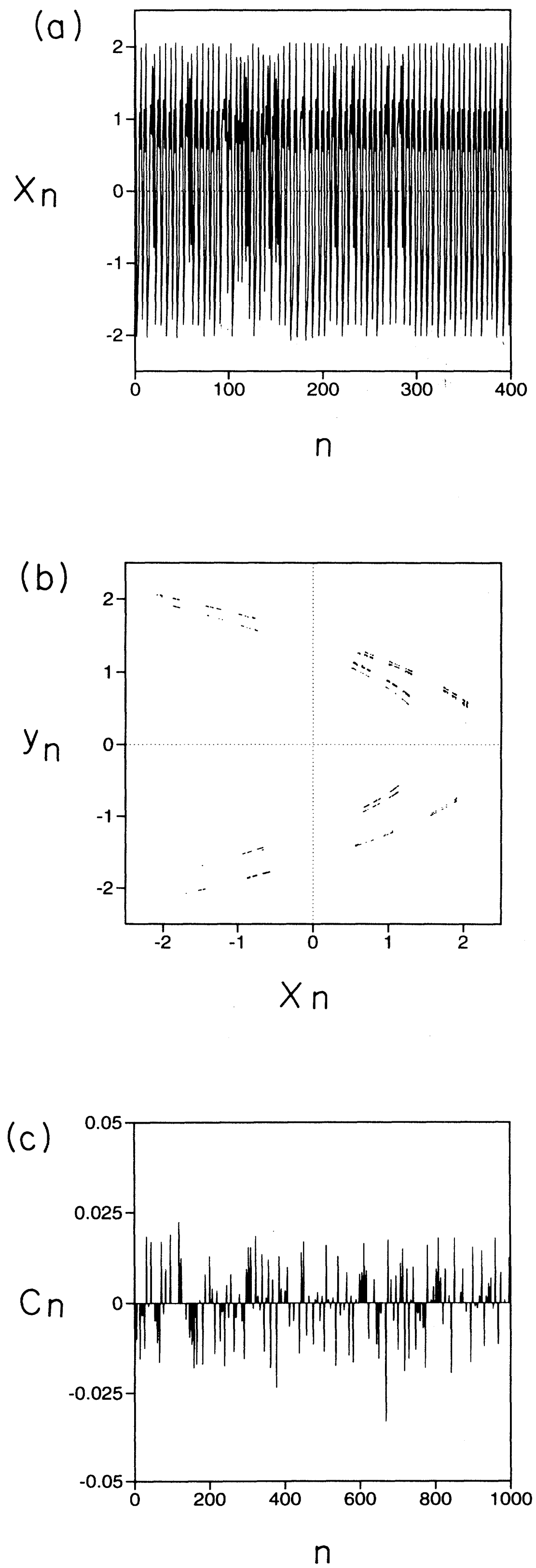

FIG. 12. Results for the Hénon map with $p=2.0$. (a) x_n versus n with control and $M=5$. (b) Sketch of the chaotic saddle with (x_n, y_n). (c) c_n versus n.

the smallest amplitude such that the orbit is still kicked out of L_M with $M=5$. Figure 12(a) shows a plot of x_n versus n with strategy A control. A rough sketch of the chaotic saddle can be obtained by plotting (x_n, y_n) [Fig. 12(b)]. For the same case, c_n versus n is displayed in Fig. 12(c). The largest value of $|c_n|$ is $c_{\max}=0.034$, and the fraction of time with $c_n \neq 0$ is 0.14.

A simple alternate strategy that can be employed here is to choose $c_n \equiv p_c - p \approx 0.6$ which permanently eliminates the situation in Fig. 11(b). The price one pays here is a much larger control that is on 100% of the time.

IV. EFFECT OF NOISE

The effect of noise is illustrated below for the crisis transition in one-dimensional maps (see Sec. II A). For simplicity we treat uniformly distributed finite amplitude noise whose impact on the map is additive. Namely, we consider

$$x_{n+1} = F_\delta(x_n, p, 0) = p x_n (1 - x_n) + \delta_n + c_n \;, \tag{8}$$

where $|\delta_n| < \Delta$. In contrast to the situation of Sec. II A, an orbit point, with the presence of noise, can enter L_i

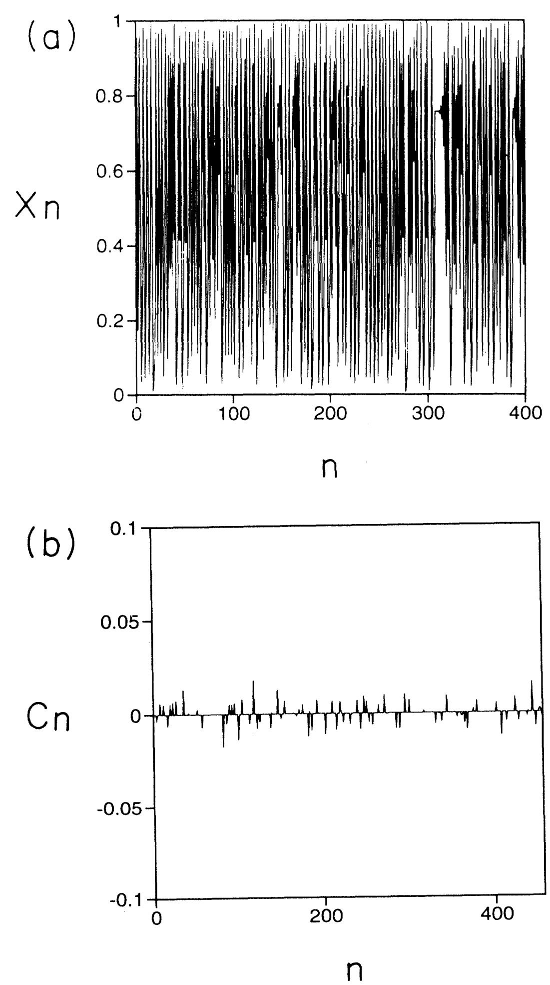

FIG. 13. Results for the noisy logistic map. (a) x_n versus n for $p=4.1$ and $M=2$ and (b) c_n versus n.

without first going through L_j, where $i<j$. To counter this, we first augment every interval in the original L_{m-1} by the amount of Δ on both sides and define the inverse image under F^{-1} of the new set of intervals to be L'_m. Similar to Eq. (4) we choose c_n so that x_{n+1} falls outside of $L'_M \cup L'_{M-1} \cup \cdots \cup L'_1 \cup L'$. Figure 13(a) shows x_n versus n with control for $p=4.1$, $\Delta=0.002$, and $M=2$, and Fig. 13(b) shows c_n versus n. The largest value of $|c_n|$ is $c_{\max}=0.018$ and the fraction of time with $c_n \neq 0$ is 0.1.

Note that, in the noisy case, M cannot be set too large because otherwise intervals in $L'_M \cup L'_{M-1} \cup \cdots \cup L'_1 \cup L'$ will begin to have overlap. This situation becomes more severe as Δ increases. In fact, for $\Delta=0.02$, we find that the simple alternate strategy in Sec. II A with noise modification works more effectively than strategy A. Specifically, if we set

$$c_n = -(p-p_c)/p_c - \Delta$$

whenever the orbit falls in L', the orbit can be controlled with $c_{\max}=0.045$, and the fraction of time with $c_n \neq 0$ is 0.13. On the other hand, if strategy A is implemented for the same conditions, one obtains $c_{\max}=0.063$ with the fraction of time where $c_n \neq 0$ to be 0.5.

V. DISCUSSION

In this paper we have dealt with low-dimensional chaotic systems. The evident biological chaos discussed in Sec. I B above may or may not be low dimensional, and this may also depend on the biological setting. At present, there is controversy as to whether low-dimensional chaos is relevant in biology [23]. Recent evidence suggests low dimensionality in some cases [16,17]. Whether or not the considerations in this paper prove relevant in biology awaits further investigation.

ACKNOWLEDGMENTS

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